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AN EXTENSION OF THE ALGEBRA OF LOGIC

BY the "algebra of logic" I mean, for the purposes of the present paper, Boole's calculus, in its classic form. I shall ignore, for the time, all the applications of this calculus to "classes," to "propositions," or to any other special sorts of objects. By the "Boolean entities" I shall here mean simply whatever entities conform to the laws which Boole's algebra expresses. There are such entities,—for instance, "classes," and "propositions," as well as "areas." But there are other Boolean entities than these; and I am here concerned only with Boolean entities abstractly viewed.

I

Considered as an algebra, the Boolean calculus, as is well known, occupies an unique but disappointing place among the algebras known to modern mathematics. In opening two remarkable contributions to the algebra of logic which were published a few years ago[1] Whitehead said that Boole's algebra might be compared with the chemical element argon. For, as Whitehead remarked, Boole's calculus had so far refused to form compounds with other equally elementary theories. Whitehead himself undertook, in the papers in question, to contribute towards the needed change of this unprofitable state of isolation. But, important as his results were, Whitehead has not, so far as I know, published further researches upon the same topic. The problem remains still on our hands: Can any one discover devices to make this argon among the pure algebras somewhat better disposed to unite with its fellow-algebras?

This question may seem to have a merely formal interest: and in this paper I shall deal wholly with formalities. Yet, as I hope to show in future papers, very interesting philosophical issues are bound up with the answer to the question which Whitehead's comparison of the Boolean calculus to argon presents to our notice. This is a region of philosophy where some of the most abstract, and some of the deepest and richest of philosophical interests lie very near together.

[1] See *American Journal of Mathematics*, Vol. 23, pages 139, 297.

II

The Nature of the Boolean Operations

The general reason why Boole's calculus has proved so austerely unproductive of the mathematical novelties for which Boole himself hoped, is well known. The fundamental operations of the Boolean calculus, *viz.*, the "addition" and the "multiplication" which characterize this algebra, appear, at first sight, to promise notable new combinations, since, like the corresponding operations in ordinary algebra, they are both commutative and associative. Furthermore, each of these Boolean operations is distributive with reference to the other. Their dual relation to "negation," as expressed in "De Morgan's theorem," is a very attractive property, which especially helps to give to this algebra its unique place amongst forms of symbolism. But alas, neither the "multiplication" nor the "addition" of this algebra is unconditionally invertible. Nor is the result of such inversion, when there is such a result, free from ambiguity. "Division" and "subtraction" have existence only subject to exasperating restrictions; and have never received any really notable development. In brief, the fundamental operations of the algebra are not group-operations. Hence the theory of groups is, except for one striking, but, so far, comparatively unfruitful exception, inapplicable to the Boolean algebra in its classic form. But without group-operations how shall an algebra progress?

III

The Operation of Jevons, the Prime-Functions of Whitehead, and the *T*-Relation

The exception just mentioned, the one group-operation which the Boolean calculus in its original form permits, was first noticed by Jevons. Schroeder and, still later, Whitehead, have dealt with it at considerable length. I have here only a word to add to the observations which Schroeder and Whitehead have made upon this particular topic; but this word relates, I believe, to a logical relation whose philosophical importance may appear, as I hope, in papers which I hope later to prepare.

Jevons noted that the three Boolean equations:—

$$a\bar{b} + \bar{a}b = c,$$
$$b\bar{c} + \bar{b}c = a,$$
$$a\bar{c} + \bar{a}c = b,$$

express precisely equivalent propositions, since each of them follows from either of the others. Schroeder and Whitehead have in somewhat different ways expressed the very natural observation that this

equivalence of the three equations of Jevons defines a group-operation.

Let $a\bar{b} + \bar{a}b = (a \circ b)$, where the symbol $\circ$ stands for an operation performed upon the elements of the pair (a, b). Then, if we suppose that this operation is viewed as already known,—we observe that whatever pair of the "logical entities" of the Boolean calculus we may choose, the entity $(a \circ b)$ is itself a perfectly determinate Boolean logical entity. The operation $\circ$ is commutative and associative. The three equations of Jevons show that this operation is not only invertible, but is also its own inverse, so that the three symbolic expressions, $(a \circ b) = c$, $(a \circ c) = b$, $(b \circ c) = a$, express mutually equivalent propositions. The group defined by this operation is the "axial group," as has been pointed out by Professor Miller, of the University of Illinois, to whom I owe this last observation.

Whitehead has proposed to call functions whose form is that of $a\bar{b} + \bar{a}b$, "prime functions" or simply "primes." For a reason, explained in his papers,[2] Whitehead also calls these functions "primary primes."

The equation: $a\bar{b} + \bar{a}b = c$ appears, at first sight, to be expressive of a triadic relation of the elements (a, b, c). But as early as 1905 I myself noted that the relation involved is in fact *tetradic*. This latter observation was at that time, I think, new; and I have never heretofore printed the very obvious statements regarding this tetradic relation which here follow, and which I have constantly used in lectures ever since 1905. Some fellow-students may still find novel the form of expression which I employ.

Let us suppose true the equation:—

$$a\bar{b} + \bar{a}b = c\bar{d} + \bar{c}d.$$

Then the four elements (a, b, c, d) stand in a symmetrical tetradic relation which can be expressed by solving the equation, successively, for a, for b, for c, and for d. Thus,

$$\begin{aligned} a &= (bc + \bar{b}\bar{c})d + (b\bar{c} + \bar{b}c)\bar{d}; \\ b &= (ad + \bar{a}\bar{d})c + (a\bar{d} + \bar{a}d)\bar{c}; \\ &= (ac + \bar{a}\bar{c})d + (a\bar{c} + \bar{a}c)\bar{d}. \end{aligned}$$

That is, in case the equation: $a\bar{b} + \bar{a}b = c\bar{d} + \bar{c}d$, is true, each of the four elements of the tetrad (a, b, c, d) is a determinate symmetrical function of the three remaining elements, while the form of this function remains the same, whatever one of the four elements we choose to express in terms of the others. I propose to symbolize the totally symmetrical tetradic relation here in question by the relational form $T(abcd)$, which is to be read as a proposition: "The four

[2] See *American Journal of Mathematics*, Vol. 23, page 147.

elements (a, b, c, d) stand in the four-term relation T." The proposition $T(abcd)$ is thus precisely equivalent to any one of the equations:

$$ab + \bar{a}b = c\bar{d} + \bar{c}d;\ a\bar{c} + \bar{a}c = b\bar{d} + \bar{b}\bar{d},\ \text{etc.}$$

The properties of this T-relation are then the following:—

1. The relation is (as just pointed out) totally symmetrical. That is, $T(abcd) = T(cdba) = T(dcab)$, etc.

2. Given (a, b, c) or, in fact, given any triad of the elements of the tetrad (a, b, c, d), chosen quite without restriction,—then, by the requirement that $T(abcd)$ shall be true, the fourth element of the tetrad is uniquely determined.

3. The T-relation $T(abcd)$ remains invariant in case, for any *two* of the elements of the tetrad, we substitute their respective negatives. Thus $T(abcd) = T(ab\bar{c}\bar{d}) = T(\bar{a}b\bar{c}d)$, etc.

4. The T-relation is *transitive by pairs*. That is, if $T(abcd)$, and if $T(abef)$, it follows that $T(cdef)$. For $a\bar{b} + \bar{a}b = c\bar{d} + \bar{c}d = e\bar{f} + \bar{e}f$.

5. If $T(abcd)$ and if $a = b$, then it follows that $c = d$; if $b = c$, then $a = d$, etc. Conversely, $T(aabb)$, $T(\bar{a}\bar{d}bb)$, $T(aaaa)$, are true propositions, whatever Boolean elements a and b may be.

If, hereupon, we make the logical element 0 a member of a tetrad, and if $T(abc0)$ is true, this fact may also be expressed by any one of the three equations of Jevons:—

$$a\bar{b} + \bar{a}b = c,\ b\bar{c} + \bar{b}c = a,\ a\bar{c} + \bar{a}c = b.$$

It follows that equations involving the "prime-functions" of Whitehead can be expressed in terms of T-relations; and that, in particular, the equation: $a\bar{b} + \bar{a}b = c$ should be regarded, *not* as simply expressive of a triadic relation of (a, b, c), but rather as expressing a symmetrical tetradic relation of $a, b, c,$ and 0.

This, the only group-operation of the classical Boolean algebra, may thus be regarded as deriving its properties from those of the T-relation. The total symmetry of this tetradic relation is responsible for the simplicity of the group in question, and for its comparative unfruitfulness as a source of novelties in the Boolean calculus.

I shall have occasion to return to the T-relation in various stages of the inquiry which is herewith begun.

IV

The Ordinal Functions

So far we have traversed what is, on the whole, decidedly familiar ground. From this point onwards we shall deal with matters which I believe to be novel.

There exists a class of functions, in the algebra of logic, to which, since 1909, when I first observed some of their more interesting properties, I have devoted a good deal of study. Here is not yet the place to tell why these functions are as important for logical theory as I believe them to be. My present task will be limited to defining these functions, and to showing that, by making a due use of them, we can easily lay the foundation for a very considerable extension of the classical Boolean calculus. I shall give to the functions in question the name "ordinal functions."

Every student of Boole's algebra is familiar with functions of the form:—

$$p = axy + bx\bar{y} + c\bar{x}y + d\bar{x}\bar{y}.$$

It occasionally happens, in dealing with the solution of equations, that one may meet with the special case where $a = \bar{d}$, $b = \bar{c}$. The foregoing function then becomes:—

$$\begin{aligned} p &= axy + bx\bar{y} + \bar{b}\bar{x}y + \bar{a}\bar{x}\bar{y} \\ &= abx + \bar{a}\bar{b}\bar{x} + a\bar{b}y + \bar{a}b\bar{y}. \end{aligned}$$

If $p = 0$, the resulting equation has an interesting pair of roots. For by virtue of the foregoing transformation, which the special function here in question permits, the two unknowns (x, y) in the equation:—

$$axy + bx\bar{y} + \bar{b}\bar{x}y + \bar{a}\bar{x}\bar{y} = 0,$$

can be separated for the purpose of solution. For we have, as above:

$$abx + \bar{a}\bar{b}\bar{x} + a\bar{b}y + \bar{a}b\bar{y} = 0.$$

Hence

$$\begin{aligned} \bar{a}b &-\!< x -\!< \bar{a} + \bar{b}, \\ \bar{a}b &-\!< y -\!< \bar{a} + b. \end{aligned}$$

The equation is therefore soluble, not only without condition, but in a peculiarly simple form.

By an ordinal function I mean a function of four terms which has the form here illustrated by p.[3]

In the foregoing case, p appears as a function of the two unknowns (x, y), and of the two coefficients (a, b), together with their negatives $(\bar{a}, \bar{b})$. For our further purposes, this distinction between the "unknowns" and "the coefficients" may be left out of account. But we still need to note that an ordinal function is to be viewed as a function of four terms which are regarded as first divided into two pairs, while the function itself depends upon the way in which these pairs are chosen, are arranged, and then are submitted to the operations which characterize the form above illustrated.

[3] The ordinal functions here defined form one special case in the class of functions which are sums or products of Whitehead's "secondary primes." But Whitehead does not discuss in any detail this special case.

A few examples may help to make clear what is essential to the form of an ordinal function. Let the four elements (a, b, c, d) and their respective negatives be used. Let

$$p = abc + \bar{a}\bar{b}\bar{c} + a\bar{b}d + \bar{a}b\bar{d},$$
$$q = a\bar{b}c + \bar{a}b\bar{c} + a\bar{c}d + \bar{a}c\bar{d},$$
$$r = b\bar{c}d + \bar{b}c\bar{d} + ab\bar{d} + \bar{a}\bar{b}d,$$
$$s = \bar{a}\bar{c}\bar{d} + acd + abc + \bar{a}\bar{b}c.$$

In the formation of each of these functions there is followed a set of rules which may be grasped by analyzing first the structure of the function p.

In order to form p we first select the two pairs (a, b) and (c, d). Let the pair (a, b) be called, for the moment, the *first* of these pairs, and the pair (c, d) the *second* pair. Each of these pairs, as it is here written, has its own first and its own second member. p is then defined by means of the five directions which here follow:

1. Take the continued product of both elements of the first pair into the first element of the second pair. We thus get the product abc.

2. Then form the product of the respective negatives of both elements of the first pair into the negative of the first member of the second pair. We now have the product $\bar{a}\bar{b}\bar{c}$.

3. Next form the continued product of the first member of the first pair, into the negative of the second member of the first pair, and into the second member of the second pair. We thus form the product $a\bar{b}d$.

4. Hereupon form the product of the negative of the first member of the first pair, into the second member of the first pair, and into the negative of the second member of the second pair. This gives us the product $\bar{a}b\bar{d}$.

5. Finally, take the sum of the four products thus successively formed. This sum gives us $g = abc + \bar{a}\bar{b}\bar{c} + a\bar{b}d + \bar{a}b\bar{d}$.

It will facilitate a survey of the rule whereby an ordinal function is formed, to use the following new symbolism. Let

$$abc + \bar{a}\bar{b}\bar{c} + a\bar{b}d + \bar{a}b\bar{d} = (ab\,;\,cd).$$

Here the expression which follows the symbol $=$ is a new symbol introduced, quite arbitrarily, simply to remind us of the foregoing five directions. This new symbol has the advantage of showing that the pairs (a, b) and (c, d) are distinguished, for the purpose here in question, as the first and the second pair. In the arbitrary symbol $(ab\,;\,cd)$, the semicolon separates these pairs. And the expressions ab, and cd, when thus separated by a semicolon, and together inserted, in this order, in a parenthesis, are not, in that context, to be understood as symbolizing logical products. The ex-

pression: $g = (ab;\ cd)$ is to be read: "The function g is that function of the four elements (a, b, c, d), which is formed by taking (a, b) and (c, d) as distinct pairs, whereof (a, b) is the first pair and (c, d) the second, and by then applying the five directions given above. Since these directions are themselves quite general, it would be easy to interpret as determinate ordinal functions any analogous expression, such as $x = (pq;\ rs)$; $y = (pr;\ qs)$, etc. Thus $x = (pq;\ rs)$ is equivalent, in terms of the ordinary Boolean calculus, to the equation:

$$x = pqr + \bar{p}\bar{q}\bar{r} + p\bar{q}s + \bar{p}q\bar{s}.$$

And $y = (pr;\ qs)$ is equivalent to the equation:—

$$\begin{aligned} y &= prq + \bar{p}\bar{r}\bar{q} + p\bar{r}s + \bar{p}r\bar{s} \\ &= pqr + \bar{p}\bar{q}\bar{r} + p\bar{r}s + \bar{p}r\bar{s}. \end{aligned}$$

As to the following functions of the tetrad (a, b, c, d):

$$\begin{aligned} h &= abd + \bar{a}\bar{b}\bar{d} + a\bar{b}\bar{c} + \bar{a}bc, \\ m &= a\bar{b}c + \bar{a}b\bar{c} + a\bar{c}d + \bar{a}c\bar{d}, \end{aligned}$$

it is plain that they may be written:

$$\begin{aligned} h &= (ab;\ dc) = (a\bar{b};\ \bar{c}d), \\ m &= (ac;\ \bar{b}d) = (a\bar{c};\ d\bar{b}). \end{aligned}$$

For if we choose any one of the four expressions here written in parentheses, and if we regard each expression as a shorthand direction to apply our rule for forming ordinals to the two pairs which the semicolon (as used in each of these cases), separates, the resulting functions are as we have just written them.

In sum then, I mean, by an ordinal function, a function of four elements such that, if we begin with the form $(ab;\ cd)$, interpreted as above, we can form all possible ordinal functions by substituting for the tetrad (a, b, c, d) other tetrads of symbols which stand for logical elements; by changing the order of the elements and of the pairs which are in question; and by introducing the negatives, $\bar{a}$, $\bar{b}$, etc., into the expressions.

In case we confine ourselves to the elements (a, b, c, d) and their respective negatives, the permutations and arrangements possible in defining the forms $(ab;\ cd)$, $(\bar{a}\bar{b};\ \bar{d}c)$ $(ca;\ db)$ $(a\bar{d};\ \bar{b}c)$, etc., appear, at first sight, too numerous for an easy survey. But there are relations between pairs of formally distinct ordinal functions which greatly simplify the task of following the variations that our definition permits. Of these relations, the most important is expressed by the symbolic equation:

$$(ab;\ cd) = (cd;\ ab).$$

In the ordinary notation this becomes:

$$abc + \bar{a}\bar{b}\bar{c} + abd + \bar{a}b\bar{d} = acd + \bar{a}\bar{c}\bar{d} + bc\bar{d} + \bar{b}\bar{c}d.$$

The verification of this latter equation requires only a very simple computation. But the property expressed is especially characteristic of the ordinal functions. In view of the solution above given in case of an equation whose left-hand member is an ordinal function, it may require but a little reflection to see that the transitivity of the illative relation, the theory of elimination, as it forms part of the theory of logical equations, and consequently the ordinary theory of the syllogism, may all of them be viewed as standing in a close relation to this fundamental principle of the theory of the ordinal functions.

In consequence of the foregoing, we may now readily verify the symbolic equations

$$(ab\,;\,cd) = (cd\,;\,ab) = (a\bar{b}\,;\,dc) = (\bar{a}\bar{b}\,;\,\bar{c}\bar{d}) = (cd\,;\,ba) = (\bar{c}\bar{d}\,;\,\bar{a}\bar{b}).$$

The negation of ordinal functions leads to interesting forms.

If

$$g = (ab\,;\,cd), \text{ then } \bar{g} = (ab\,;\,\bar{c}\bar{d})\bar{} = (cd\,;\,\bar{a}\bar{b}).$$

For

$$g = abc + \bar{a}\bar{b}\bar{c} + a\bar{b}d + \bar{a}b\bar{d};$$

and therefore,

$$\bar{g} = ab\bar{c} + \bar{a}\bar{b}c + a\bar{b}\bar{d} + \bar{a}bd;$$

while, by the foregoing,

$$g = acd + \bar{a}\bar{c}\bar{d} + bc\bar{d} + \bar{b}\bar{c}d;$$

and therefore,

$$\bar{g} = \bar{a}cd + a\bar{c}\bar{d} + \bar{b}cd + b\bar{c}d.$$

When we study the ordinal functions which may be derived from $(ab\,;\,cd)$ by changing the order of the elements or of the pairs, and by introducing, in various possible ways, the symbol for negation, we find cases where two such functions are not, in general, equivalent, but become so upon condition that the elements of the tetrad (a, b, c, d) stand in some definite relation to one another. Thus let $g = (ab\,;\,cd)$, while $h = (ba\,;\,dc)$.

Then

$$g = abc + \bar{a}\bar{b}\bar{c} + a\bar{b}d + \bar{a}b\bar{d},$$
$$h = abd + \bar{a}\bar{b}\bar{d} + a\bar{b}\bar{c} + \bar{a}bc = (ab\,;\,d\bar{c}).$$

Hence,

$$\bar{g} = ab\bar{c} + \bar{a}\bar{b}c + a\bar{b}\bar{d} + \bar{a}bd = (ab\,;\,\bar{c}\bar{d}),$$
$$\bar{h} = ab\bar{d} + \bar{a}\bar{b}d + a\bar{b}c + \bar{a}b\bar{c} = (ab\,;\,dc).$$

In consequence,

$$g\bar{h} + \bar{g}h = abc\bar{d} + ab\bar{c}d + \bar{a}\bar{b}\bar{c}\bar{d} + \bar{a}\bar{b}cd + a\bar{b}\bar{c}\bar{d} + a\bar{b}cd + \bar{a}b\bar{c}\bar{d} + \bar{a}bcd + \bar{a}b\bar{c}d.$$

Or, more briefly stated,

$$g\bar{h} + \bar{g}h = (ab + \bar{a}\bar{b})(c\bar{d} + \bar{c}d) + (a\bar{b} + \bar{a}b)(cd + \bar{c}\bar{d}).$$

If $g = h$, then both the first and the second members of this equation reduce to zero. In that case we have: $a\bar{b} + \bar{a}b = c\bar{d} + \bar{c}d$. This last expression is equivalent to asserting $T(abcd)$.

Therefore, if we transform the ordinal $g = (ab;\ cd)$ into the ordinal $(ba;\ dc) = h$, by separately reversing the order of the members of the first and the second pairs of the symbol for g, while leaving all else unchanged, then the ordinals g and h can be mutually equivalent if, and only if, $T(abcd)$. The T-relation thus stands in close connection with the properties of the ordinal functions.

These few cases serve to give a first glimpse of the decidedly interesting properties of the ordinal functions. I omit further developments at this stage of the inquiry; but I hope to have more to say about the ordinals in a subsequent paper.

V

Directed Pairs, and an Operation upon Such Pairs

We are now ready to define an operation which is based upon the operations of the ordinary algebra of logic, but which is applied to a new system of logical entities.

These new entities are *pairs* of Boolean logical elements. Just as, in the familiar modern theory of the rational numbers, a rational number is defined as a pair of whole numbers, so, in the theory to which I now pass, the entities to be considered are not Boolean elements, such as a, b, c, etc., but are pairs of such elements. And precisely as, in the modern theory of the rational numbers, the pairs of natural numbers which are in question are *directed* pairs, each pair having its first and its second member, or its upper and its lower member, so, in the following theory of logical pairs, I shall deal with directed logical pairs. Let the symbol a/b, or $\frac{a}{b}$, be used for the directed pair consisting of the elements $(a,\ b)$, the element a being taken as the first, and b as the second member.

Our foregoing sketch of the properties of the ordinals has called our attention to the pairs of elements which are used in defining such an ordinal as $g = (ab;\ cd)$. But the pairs of which we made mention in our foregoing account of the ordinals, were not treated *as* pairs. The elements used entered, in a determinate way, into the definition of g. But g was itself a Boolean element. The functions in questions were Boolean functions. The elements a, b, c, etc., were treated only in so far as they were subjected to the operations of the Boolean calculus. The symbol $(ab;\ cd)$ was itself a mere shorthand for expressing the rule whereby the formation of an ordinal function was guided.

But from this point onwards we are to deal with our directed pairs as new entities, and are to subject them to new operations.

VI

The Ordinal Pair-Operation

Let a/b, c/d be the symbols, respectively, of the two directed pairs (a, b), (c, d). The form a/b means simply that a is taken as the first, and b as the second member of the pair (a, b), for the purpose of the operations into which these pairs are to enter. The symbol c/d has an analogous interpretation in case of the pair (c, d). This adoption of the symbols which are ordinarily used for fractions has in this case *merely* the significance which is given to it by the definition just stated.

Now the two directed pairs a/b and c/d may be so combined as to define, uniquely, a new pair. This pair shall be defined as the directed pair of ordinals g/h, when $g=(ab\,;\,cd)$ and $h=(ba\,;\,dc)$. Let the combination in question be viewed as an operation upon the pairs a/b, c/d. Symbolize this operation by o. Then

$$\frac{a}{b}\,\mathrm{o}\,\frac{c}{d}=\frac{g}{h}=\frac{(ab\,;\,cd)}{(ba\,;\,dc)}.$$

It is plain that the directed pair g/h is uniquely determined by the directed pairs a/b, c/d.

Since $(ab\,;\,cd)=(cd\,;\,ab)=g$, while $(ba\,;\,dc)=(dc\,;\,ba)=h$, the operation o, as now defined, is commutative. That is, $a/b\,\mathrm{o}\,c/d=c/d\,\mathrm{o}\,a/b$.

Hereupon, we come to that feature of the new operation which constitutes the first contrast by which it is distinguished both from the addition and from the multiplication of the ordinary Boolean calculus. The new operation, namely, is *invertible*. That is, given the pair g/h, and either of the pairs a/b or c/d, the other of these pairs is uniquely determined.

For from the equations

$$\begin{aligned} g&=abc+\bar{a}\bar{b}\bar{c}+a\bar{b}d+\bar{a}b\bar{d}\\ &=acd+\bar{a}\bar{c}\bar{d}+bc\bar{d}+\bar{b}\bar{c}d,\\ h&=abd+\bar{a}\bar{b}\bar{d}+a\bar{b}\bar{c}+\bar{a}bc\\ &=bcd+\bar{b}\bar{c}\bar{d}+\bar{a}c\bar{d}+a\bar{c}d \end{aligned}$$

we can deduce, by an easy computation, the consequences:—

$$\begin{aligned} a&=cdg+\bar{c}\bar{d}\bar{g}+c\bar{d}\bar{h}+\bar{c}dh=(cd\,;\,g\bar{h})=(dc\,;\,gh),\\ b&=cdh+\bar{c}\bar{d}\bar{h}+c\bar{d}g+\bar{c}d\bar{g}=(cd\,;\,hg)=(hg\,;\,cd),\\ c&=abg+\bar{a}\bar{b}\bar{g}+a\bar{b}\bar{h}+\bar{a}bh=(ab\,;\,g\bar{h})=(ba\,;\,gh),\\ d&=abh+\bar{a}\bar{b}\bar{h}+a\bar{b}g+\bar{a}b\bar{g}=(ab\,;\,hg)=(hg\,;\,ab). \end{aligned}$$

The interest of these facts is greatly increased by the consideration

that the invertibility of our operation is possible universally and without restriction, and can be expressed in a peculiarly simple form. Thus, if

$$a/b \circ c/d = g/h,$$

it is easy to show that

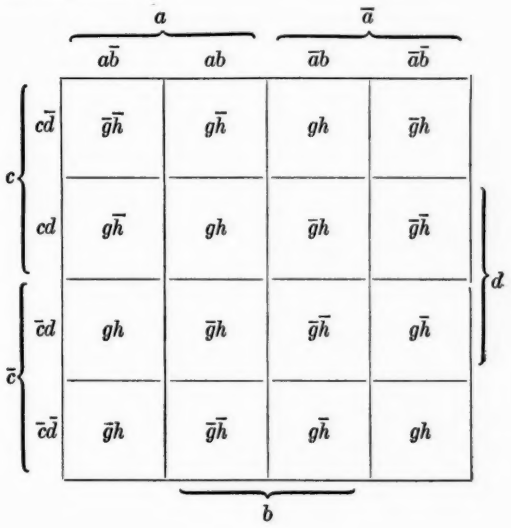
$$g/h \circ b/a = c/d,$$

while

$$g/h \circ d/c = a/b.$$

This, in fact, is precisely what is formulated by means of the four equations written above, and expressive of the values of a, b, c, and d.

The novelty and simplicity of the considerations here involved, make it worth while to express the facts in the form of what may be called a "Newlin-diagram,"[4] which brings before the eye the divisions of a universe of discourse containing four independent terms. It is well to remember, however, that any such diagram expresses a special application of the Boolean calculus, while the laws here in question hold true of the pure algebra, and are independent of all applications to "classes," to "areas," or to other special sorts of entities.



Let

$$g = abc + \bar{a}b\bar{c} + abd + \bar{a}b\bar{d} = (ab;\ cd),$$
$$h = abd + \bar{a}\bar{b}d + a\bar{b}\bar{c} + \bar{a}bc = (ba;\ dc).$$

The inverse operation just noted can easily be read off from this diagram. The formal analogy of our new operation, and of its inverse, to the multiplication and division of ordinary rational numbers, is, as far as it goes, worthy of notice. In case of the rational

[4] See the article by Professor W. J. Newlin, in this JOURNAL, Vol. III., page 539.

numbers, to divide by a rational number is equivalent to multiplying by the inverse, that is, by the reciprocal of the pair of whole numbers which constitutes the divisor. Our pairs of Boolean entities obey, with respect to our new operation, an analogous rule. The inverse of our operation is the direct operation with one of the factors of the original operation inverted, and then combined with the result of the original operation.

VII

The Associative Property of the Ordinal Pair Operation

We next come to a still more important fact. Our new operation is not only commutative and invertible, but also associative. That is, if we have given to us the three independent pairs a/b, c/d, e/f, then

$$\left[\frac{a}{b}\circ\frac{c}{d}\right]\circ\frac{e}{f}=\frac{a}{b}\circ\left[\frac{c}{d}\circ\frac{e}{f}\right].$$

The computations needed to establish this property of our operation are of necessity a little diffuse for complete statement in so summary a paper as the present one. It may be worth while, however, to give an outline of what seems to be a convenient mode of dealing with the matter, leaving to the reader the verification of the details of the computation, if he chooses to work them out. It may be remarked that, by means of a six-term "Newlin-diagram," or better by means of a pair of such diagrams, each diagram presenting to the eye one of the associations of pairs which is in question, the associative property of our operation can be made visible. But a six-term Newlin-diagram is somewhat troublesome to print.

It is worth noting, and is easily verifiable, that, if $a/b\circ c/d=g/h$, then

$$\begin{aligned}
gh &= abcd+\bar{a}bc\bar{d}+a\bar{b}cd+\bar{a}\bar{b}c\bar{d}=(ad+\bar{a}\bar{d})(bc+\bar{b}\bar{c}),\\
g\bar{h} &= abc\bar{d}+ab\bar{c}d+\bar{a}b\bar{c}\bar{d}+\bar{a}\bar{b}\bar{c}\bar{d}=(ac+\bar{a}\bar{c})(b\bar{d}+\bar{b}d),\\
\bar{g}h &= a\bar{b}\bar{c}\bar{d}+ab\bar{c}d+\bar{a}bcd+\bar{a}\bar{b}cd=(a\bar{c}+\bar{a}c)(bd+\bar{b}\bar{d}),\\
\bar{g}\bar{h} &= a\bar{b}cd+ab\bar{c}\bar{d}+\bar{a}b\bar{c}d+\bar{a}\bar{b}c\bar{d}=(b\bar{c}+\bar{b}c)(a\bar{d}+\bar{a}d).
\end{aligned}$$

All this may be read off directly from the Newlin diagram printed above. Hereupon we may set $g/h\circ e/f=r/s$.

That is,

$$\left(\frac{a}{b}\circ\frac{c}{d}\right)\circ\frac{e}{f}=\frac{r}{s}.$$

Now, by the definition of our operation,

$$\frac{g}{h}\circ\frac{e}{f}=\frac{r}{s}=\frac{ghe+\bar{g}\bar{h}\bar{e}+g\bar{h}f+\bar{g}h\bar{f}}{ghf+\bar{g}\bar{h}\bar{f}+gh\bar{e}+\bar{g}he}.$$

Substituting, in the right-hand member of this equation, the values of gh, $g\bar{h}$, $\bar{g}h$, and $\bar{g}\bar{h}$, as given above, we find:

$$r=(ad+\bar{a}\bar{d})(bc+\bar{b}\bar{c})e+(a\bar{d}+\bar{a}d)(b\bar{c}+\bar{b}c)\bar{e}$$
$$+(ac+\bar{a}\bar{c})(b\bar{d}+\bar{b}d)f+(a\bar{c}+\bar{a}c)(bd+\bar{b}\bar{d})\bar{f},$$
$$s=(ad+\bar{a}\bar{d})(bc+\bar{b}\bar{c})f+(a\bar{d}+\bar{a}d)(b\bar{c}+\bar{b}c)\bar{f}$$
$$+(a\bar{c}+\bar{a}c)(bd+\bar{b}\bar{d})e+(ac+\bar{a}\bar{c})(b\bar{d}+\bar{b}d)\bar{e}.$$

But the directed pair r/s is, as we have defined it, the equivalent of

$$\left(\frac{a}{b}\circ\frac{c}{d}\right)\circ\frac{e}{f}.$$

Hereupon let us set

$$\frac{c}{d}\circ\frac{e}{f}=\frac{m}{n},$$

and suppose

$$\frac{a}{b}\circ\frac{m}{n}=\frac{u}{v}.$$

Then, by our definition,

$$\frac{a}{b}\circ\left(\frac{c}{d}\circ\frac{e}{f}\right).$$

We have also

$$\frac{m}{n}=\frac{(cd\,;\,ef)}{(dc\,;\,fe)}=\frac{cde+\bar{c}\bar{d}\bar{e}+c\bar{d}f+\bar{c}d\bar{f}}{cdf+\bar{c}\bar{d}\bar{f}+c\bar{d}\bar{e}+\bar{c}de}.$$

Furthermore

$$\frac{a}{b}\circ\frac{m}{n}=\frac{u}{v}.$$

Hence

$$u=abm+\bar{a}\bar{b}\bar{m}+a\bar{b}n+\bar{a}b\bar{n}.$$

Substituting the values of m and of n, respectively, as defined above, we discover hereupon that

$$u=ab(cde+\bar{c}\bar{d}\bar{e}+c\bar{d}f+\bar{c}d\bar{f})$$
$$+a\bar{b}(cdf+\bar{c}\bar{d}\bar{f}+c\bar{d}\bar{e}+\bar{c}de)$$
$$+\bar{a}b(cd\bar{f}+\bar{c}\bar{d}f+c\bar{d}e+\bar{c}d\bar{e})$$
$$+\bar{a}\bar{b}(cd\bar{e}+\bar{c}\bar{d}e+c\bar{d}\bar{f}+\bar{c}df).$$

Rearranging this expression for the value of u, we find:

$$u=e(ad+\bar{a}\bar{d})(bc+\bar{b}\bar{c})+\bar{e}(a\bar{d}+\bar{a}d)(b\bar{c}+\bar{b}c)$$
$$+f(ac+\bar{a}\bar{c})(b\bar{d}+\bar{b}d)+\bar{f}(a\bar{c}+ac)(bd+\bar{b}\bar{d}).$$

Comparing this result with the value above given for r, we find that $r=u$.

By a precisely analogous computation we find that $s=v$, and thus we reach the result that:

$$\left(\frac{a}{b}\circ\frac{c}{d}\right)\circ\frac{e}{f}=\frac{a}{b}\circ\left(\frac{c}{d}\circ\frac{e}{f}\right)=\frac{r}{s}=\frac{u}{v}.$$

Our operation is therefore not only invertible and commutative,

but also associative. Accordingly, with respect to our new operation, the directed pairs of Boolean entities constitute an Abelian group.

VIII

Results

The properties of this group of Boolean pairs are sufficiently remarkable to deserve a summary statement. Each of the following propositions can easily be verified. The computations involved are, on the basis of the foregoing, all of them exteremely simple. They are also, so far as I know, novel.

The system of pairs now defined possesses a unit. This unit is the pair 1/1. For, by definition,

$$\frac{a}{b} \circ \frac{1}{1} = \frac{ab1 + \bar{a}\bar{b}\bar{1} + a\bar{b}1 + \bar{a}b\bar{1}}{ab1 + \bar{a}\bar{b}\bar{1} + a\bar{b}\bar{1} + \bar{a}b1} = \frac{ab + a\bar{b}}{ab + \bar{a}b} = \frac{a}{b}.$$

That is, whatever pair a/b be combined with the pair 1/1, is left invariant by that combination under the rules of our operation.

Owing to the formal analogy between our ordinal pair-operation and the multiplication of rational numbers (that analogy, namely, of which we made mention above, in speaking of the inverse of our operation), we may treat our ordinal pair-operation as a multiplication, although I have, at present, no addition-operation to set side by side with it. Regarding our operation, then, as a multiplication, we may write:

$$\frac{a}{b} \circ \frac{a}{b} = \left(\frac{a}{b}\right)^2.$$

We may use similar expressions for higher powers of a/b, and speak of cubes, etc. Only here we quickly find ourselves limited by the interesting further group-properties of our pair-operation which may next be stated.

We have, namely:

$$\left(\frac{a}{b}\right)^2 = \frac{a}{b} \circ \frac{a}{b} = \frac{aba + \bar{a}\bar{b}\bar{a} + a\bar{b}b + \bar{a}b\bar{b}}{abb + \bar{a}\bar{b}\bar{b} + a\bar{b}\bar{a} + \bar{a}ba}.$$

Hence

$$\left(\frac{a}{b}\right)^2 = \frac{ab + \bar{a}\bar{b}}{ab + \bar{a}\bar{b}}.$$

The square of any pair is, therefore, a pair consisting of equal terms. Each of these equals is a "prime-function," *viz.*, the function, $ab + \bar{a}\bar{b}$, of the members of the pair.

We therefore have, for the cube of any pair, the expression:

$$\left(\frac{a}{b}\right)^3 = \frac{ab + \bar{a}\bar{b}}{ab + \bar{a}\bar{b}} \circ \frac{a}{b} = \frac{(ab + \bar{a}\bar{b})a + (a\bar{b} + \bar{a}b)\bar{a}}{(ab + \bar{a}\bar{b})b + (a\bar{b} + \bar{a}b)\bar{b}} = \frac{ab + \bar{a}b}{ab + a\bar{b}} = \frac{b}{a}.$$

That is to say, the cube of any pair is the inverse of that pair. Or, if the inverse of our "multiplication" be regarded, for the present purpose, as a "division," we now observe that to "multiply" by the cube of any pair is equivalent to multiplying by the inverse of that pair, or is, in other words, equivalent to dividing by that pair. In still another expression our result is that, if $a/b \circ c/d = g/h$, then $(a/b)^3 \circ g/h = c/d$; while $g/h \circ (c/d)^3 = a/b$. We next proceed to the fourth powers of pairs. We have:

$$\left(\frac{a}{b}\right)^4 = \frac{a}{b} \circ \left(\frac{a}{b}\right)^3 = \frac{a}{b} \circ \frac{b}{a} = \frac{1}{1}.$$

The last of these equations is reached as follows:

$$\frac{a}{b} \circ \frac{b}{a} = \frac{abb + \bar{a}\bar{b}\bar{b} + a\bar{b}a + \bar{a}b\bar{a}}{aba + \bar{a}\bar{b}\bar{a} + a\bar{b}\bar{b} + \bar{a}bb} = \frac{ab + \bar{a}\bar{b} + a\bar{b} + \bar{a}b}{ab + \bar{a}\bar{b} + a\bar{b} + \bar{a}b}.$$

Hereupon, we observe that

$$\left(\frac{a}{b}\right)^5 = \left(\frac{a}{b}\right)^4 \circ \frac{a}{b} = \frac{a}{b} \circ \frac{1}{1} = \frac{a}{b}.$$

Thus the fourth power of every pair is the unit pair, while the fifth power of each pair is identical with the pair itself. The "period" or "order" of our pair-operation is five. If we conceive the system of directed Boolean pairs as transformed within itself by combining each pair with itself by means of the pair-operation, and by then passing to the higher powers, the first such transformation substitutes for each pair its square, a determinate pair of equals, whose members are each of them a determinate "prime-function" of the original pair. Next, the "cubes," which are the inverses of the original pairs, are produced. The next such transformation substitutes for each and all the pairs the unit pair. Combining this unit pair with each member of the original system leaves that system invariant.

IX

Modulus Pairs

The pair 1/1, the unit pair, may be called the modulus of our ordinal pair-operation. It is evident that, whatever element a may be, the equation $(a/a)^2 = 1/1$ is always true. That is, the modulus is the square of any pair that consists of equal Boolean elements.

Our system of directed pairs contains, however, other pairs which have the properties of moduli; for each such modulus pair may be regarded as the unit of an ordinal pair-operation whose group is the same as the one which we have just been studying, and whose properties are precisely analogous to those of the operation which we have been studying, so that all these operations are variations of a single one. The situation is briefly to be summed up as follows:

Let us consider the four pairs 1/1, 0/0, 1/0, 0/1, in their relations to one another, and to the other pairs of our system.

It is easy to show that

(*a*) $$\left(\frac{0}{1}\right)^2 = \frac{0}{0},$$

(*b*) $$\left(\frac{1}{0}\right)^2 = \frac{0}{0},$$

(*c*) $$\left(\frac{0}{0}\right)^2 = \frac{1}{1},$$

(*d*) $$\left(\frac{1}{1}\right)^2 = \frac{1}{1},$$

(*e*) $$\frac{1}{1} \circ \frac{0}{0} = \frac{0}{0},$$

(*f*) $$\frac{1}{0} \circ \frac{0}{1} = \frac{1}{1},$$

(*g*) $$\left(\frac{1}{0}\right)^3 = \frac{0}{1}.$$

And thus the mutual relations of the four modulus elements are stated.

But when we combine a pair a/b with each of the four moduli in succession, we get the following results:

(1) $$\frac{a}{b} \circ \frac{1}{1} = \frac{a}{b} \text{ whose cube is } \frac{b}{a},$$

(2) $$\frac{a}{b} \circ \frac{0}{0} = \frac{\bar{a}}{\bar{b}} \text{ whose cube is } \frac{\bar{b}}{\bar{a}},$$

(3) $$\frac{a}{b} \circ \frac{0}{1} = \frac{\bar{b}}{a} \text{ whose cube is } \frac{a}{\bar{b}},$$

(4) $$\frac{a}{b} \circ \frac{1}{0} = \frac{b}{\bar{a}} \text{ whose cube is } \frac{\bar{a}}{b}.$$

If, hereupon, we ask what directed pairs can be formed from a given pair a/b, by considering the four Boolean elements a, $\bar{a}$, b, $\bar{b}$, and by treating their various pairs as directed pairs, we see that the foregoing table of eight directed pairs contains all of the possible combinations, and shows how all the eight can be formed from any one of their number by using the two operations of applying the four moduli, and of raising to the third power.

But the processes in question can be greatly simplified by considering that all the four moduli can be derived from a single one of their number, by merely using our ordinal pair operation. The modulus chosen for this purpose may be either 0/1 or 1/0, at

pleasure. Thus, if we begin with 0/1, we derive the other moduli simply by considering the powers of 0/1. For we have:

$$\left(\frac{0}{1}\right)^2 = \frac{0}{0};\ \left(\frac{0}{1}\right)^3 = \frac{1}{0};\ \left(\frac{0}{1}\right)^4 = \frac{1}{1}.$$

Starting with any pair a/b, and with the single modulus 0/1, we can therefore form all the derivative pairs $\bar{a}/b$, $a/\bar{b}$, etc., merely by repeating the processes of combining with the modulus, and of raising to powers.

It is possible, however, to define a new operation such that one of the moduli, say 0/1, is the unit pair of this operation. The latter will then be derived from (and in essence equivalent to) our present ordinal pair-operation. Let us use $\cup$ as the symbol of the new operation whereof 0/1 is to be the unit pair. That is, let us require an ordinal pair-operation $\cup$ to be defined such that $a/b \cup 0/1 = a/b$ whatever pair a/b may be. To this end we have only to define $\cup$ by the equation

$$\frac{a}{b} \cup \frac{c}{d} = \left(\frac{a}{b} \circ \frac{c}{d}\right) \circ \frac{1}{0}.$$

For then

$$\frac{a}{b} \cup \frac{0}{1} = \frac{a}{b} \circ \frac{0}{1} \circ \frac{1}{0} = \frac{a}{b} \circ \frac{1}{1} = \frac{a}{b}.$$

The new operation will be so related to the old that, if $a/b \circ c/d = g/h$, then, by definition, $a/b \cup c/d = g/h \circ 1/0 = h/g$.

It is plain that by the use of the modulus element, and by raising to powers, all the results of the new operation $\cup$ can be stated in terms of our foregoing operation $\circ$, and conversely. The only novelty of the operation $\cup$ will, therefore, depend upon its choice of one of the modulus elements as its special unit.

The four moduli of our system of directed pairs are themselves pairs, and are not ordinary Boolean elements. They serve to give to the whole system properties that I believe to be not only of interest in themselves, but of no small promise for the future. In any case, here is a definite extension of the Boolean calculus, and a definite and new introduction of group-theory into this realm of the algebra of logic.

I must leave to later papers the lessons to which our study points the way. They will concern questions which I believe to be of wide philosophical bearing.

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DISCUSSION

An Answer to Professors Shotwell and Hocking

THERE appeared a short time ago in these pages[1] two reviews of my recent book,[2] by Professors J. T. Shotwell and W. E. Hocking, respectively. The former was evidently interested chiefly in the half of the book dealing with origins, and the latter chiefly in the half treating of the future of religion and of the relation of psychology to theology and metaphysics. I am indebted to these gentlemen for their appreciative comments and helpful criticisms. In several instances, however, I have not been able to yield to their arguments.

Professor Shotwell regrets that I cut off the evolution of gods from what he terms its "protoplasmic base," namely, *mana*, and he affirms that in the literature of ancient Rome one can actually trace *mana* passing into gods. My reading of W. Warde Fowler, to whom I am referred as to one offering the proof of that assertion, has not convinced me that *mana* as such passes directly into gods. That which seems to me established is that before they were personal divinities, the numena of Rome were impersonal forces. The data used by Fowler makes it further highly probable—were I better informed I might have to say "certain," instead of "probable"—that functions originally discharged by the impersonal numena, or natural phenomena in which this power found expression, came to be referred to a personal power, and this without change of name. With these two conclusions my book is in agreement. I have insisted upon the probable antecedence of the belief in impersonal power or powers, and I have put down the personification of natural events as one of the possible sources of the belief in unseen personal beings.

My knowledge of Roman anthropology is very incomplete, but I may be right in thinking that the old Romans believed in the existence, by the side of their impersonal "deities," of spirits immanent in objects and conceived of as anthropopathic, if not as anthropomorphic.[3] I do not know that these spirits have been traced back to *mana*. Are they not, some of them at least, human, souls, ghosts, originating in the observations of dreams, swoons, echoes, and like phenomena? And, if it should be said that these spirits were not gods and did not become gods, whereas the numena were or became gods, I should retort that even though it should be so among the Romans, it may have been otherwise among other peoples. There are facts which lend support to this opinion.

[1] This JOURNAL, Vol. X., pages 326 and 328.

[2] "A Psychological Study of Religion: Its Origin, Function and Future."

[3] See Fowler, "The Religious Experience of the Roman People," page 145.

Whatever may be true of the Romans need not be true of all other peoples. One must learn to bear in mind in the discussion of social origins that if individuals differ from each other in far-reaching particulars, the same is true of peoples. Different contingencies and peculiar endowments lead in religion to diversities of origin and of form. Whatever may be the truth concerning the origin of Roman divinities, it has not been shown, so far as I know, that ancestral gods such as the Chinese, Japanese, and Hebrews worshiped from time immemorial, or the Heroes to which the Greeks offered a cult anterior to Draco's Laws, have been traced back to *mana*. These gods proceeded directly from human beings.

I did not affirm that all the unseen personal powers are deities, *i. e.*, objects of religious worship. In a chapter entitled "The Making of Gods and the Essential Characteristics of a Divinity," I attempted to show why some unseen beings became gods and why others remained outside organized religion. I am sorry that Professor Shotwell should not have thought this chapter adequate. I regret also that he should persist in holding the opinion that "the one common denominator of religion is mystery." The first two chapters of the book under review show, I still think, that that opinion should be set aside. I like to believe that the reviewer has passed lightly over the second chapter. The element of mysteriousness is a *usual*, but not a *necessary* ingredient of religion; and it is found also outside of religion, for instance, in magic. But even though it were the exclusive and constant accompaniment of every pulse of religious life, it would not be its most distinctive characteristic. Religion is a form of behavior, aiming at the gratification of impulses and desires, in a specific way. It is this specific way, and not awe or mysteriousness, which is the characteristic distinctive of religion.

Professor Hocking's review is one that might well gratify even an exacting author. I can not, however, accept his main criticism. It is directed against the rôle I ascribe to psychology in providing the proof of the objective existence of the God of Christianity. After establishing (pp. 207–268) by means of a large number of quotations drawn from representative theologians that the strongest contemporary theological movement wishes to base theology upon facts of "inner experience," and claims for theology independence from metaphysics, I say, under the heading, "The Task of Psychology in the Study of Religious Life": "It is conceivable that in accomplishing this task the psychologist may encounter phenomena transcending what he can explain by the causes already known. So-called premonitions, clairvoyance, telepathy, sudden moral conversions, and mystic illuminations might, for instance, baffle his

efforts at explanation. And it might be claimed that the course of historical events testifies to a divine action" (p. 269). That the empirical theologians hold (that is a necessary part of their system) to the intervention of God into naturally determined phenomena, I had already shown sufficiently clearly. I added, nevertheless: "Dr. Reinhold Seeberg, professor of theology at the University of Berlin, maintains that certain facts of history 'certify to man the operative presence of God'; 'these facts are *distinct from the common or regular connection of events.* They are in themselves *characteristically marvelous,* or they are so constituted that man is through them made aware of the presence of God, and in them feels the power of God.'" Another professor of theology whom I have also quoted (p. 241) teaches that *"God can excite new centers of association of ideas, can arrest old associations."* When confronted by these affirmations, a question arises: "Who are those competent to determine whether an action such as these men believe possible has actually taken place?" My answer was, "it is for science [psychology] to show that any one of these possibilities has become at a particular time a reality" (p. 270). I might have added that the task of separating the "naturally" from the "divinely" conditioned is difficult enough to tax the best equipped psychologist.

Professor Hocking must have, it seems, somewhat lost sight of my argument, otherwise he would not have written: "Does the author seriously mean that we should refer to God those mental states which psychology at any time fails to account for? Is God supposed, then, to do only what natural law leaves undone? Are we not claiming too little for our science of psychology? I would rather say that when we regard our own mind as a succession of "states" having "causes," then everything in it without exception must belong to psychology: there is no conceivable residue which the science could hand over to anything except to its own unfinished investigations. Give all to psychology, and then let us be clear enough to see that the question whether a God is at work *in* those natural laws, or whether these laws in some mysterious fashion are working themselves, has not been so much as touched on."

I agree completely with Professor Hocking: psychology "can say nothing of total and ultimate causes." Nowhere in my book have I expressed a different opinion. If my reviewer thinks otherwise, it is because he forgets the pregnant distinction which it is one of the main purposes of Chapter XI. to make, namely, the distinction between the Absolute Reality and the gods of the religions, and in particular that of Christianity. The God of Professor Hocking can not be "located" by psychology; but that Being is not the God addressed in the Christian books of devotion. Whereas, the God in

whom the theologians I have quoted believe, can, by definition, be "located" in human experience; but that God is denied by Absolute Idealism. The historical religions would all cease to exist if their anthropopathic gods were replaced by the Metaphysical Absolute.[4] The affirmation of Bradley may not be gainsaid: the historical religions demand "a person in the sense of a self, amongst and over against other selves, moved by personal relations and feelings toward these others—feelings and relations which are altered by the conduct of others. And, for their purpose, what is not this, is really nothing. . . . Of course, for us to ask seriously if the Absolute can be personal in such a way, would be quite absurd." I do not know that Professor Hocking is of another mind. I think not. He will, in any case, admit that as I was not speaking of the divine Reality of Metaphysics, but of a sympathetic Being *known through his direct action in consciousness,* I was not concerned with "total ultimate causes," but only with the gods of religion, *i. e.,* beings arising from an induction upon particular parts of the "inner life," and therefore entirely within the domain of science.

To believe in God's existence because of a particular emotion, or of a particular idea or vision, is a mental process of the same species as the one which leads certain persons to the belief in spirits when they hear rappings in the wall, or when they are favored with "apparitions." Beliefs *thus produced* fall under the criticism of psychology.

If theologians turn away from metaphysics, it is because metaphysics does not provide them with the God they desire, a God making possible the Christian religion. One of my purposes in the chapter mentioned was to force upon them one or the other horn of the dilemma in the presence of which they find themselves: either return to metaphysics from which you can not get what you want, or let science do the work which you are hardly willing to perform, and abide by the consequences.

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REVIEWS AND ABSTRACTS OF LITERATURE

Thought and Things, or Genetic Logic. Vol. III. *Real Logic* (Part I.). *Interest and Art.* JAMES MARK BALDWIN. London: George Allen and Company; New York: The Macmillan Company. 1911. Pp. xvi + 284.

This is a continuation of the author's study of logic from the knower's point of view. It is the genesis of "objects" and of the forms of our

[4] See pages 245–255.

knowledge of objects, that the work attempts to describe. The first volume was entitled "Functional Logic, or Genetic Theory of Knowledge," and the second volume, "Experimental Logic, or Genetic Theory of Thought." In the first volume the author deals with what he terms the pre-logical cognitive functions, and in the second with the logical cognitive functions. In the present volume the conclusions of the other two receive restatement in Parts I. and II. "with a view to their bearing on the problem of reality"; the "logic" of "affective" experience is discussed under the title, "The Logic of Practice," in Part III.; "Aesthetic Experience" is discussed in Part IV.; the "Modes of Immediacy" are presented in Part V.; while a further brief part proposes the new word Pancalism (from the motto of the work as a whole, *τὸ καλὸν πᾶη*) as a name for the author's philosophy and projects the programme of another volume, yet to appear, which will complete the work.

In brief outline the argument of this third volume is as follows: We begin with the dualism (emphasized in Volume II.) of the actual and the imaginative, content and control, knowledge and semblance,—"this is the universal and ever-present contrast in the meanings of cognition." Every actual "thing" must have both renderings. "Belief motives make-believe and make-believe engenders belief." The imaginative rendering is always instrumental to the actual and the true. "We make-believe in order that we may believe." Truths are simply confirmed imaginings, established prospectings. "The two controls (the inner and the outer) are now adjusted to each other through the mediation of ideas or thoughts." The instrumental is therefore not truth, but the imagination of something that may become truth. When we say that truth itself is instrumental, "we are not careful enough." Imaginative constructions, on the other hand, are personal and prospective, embodying a new moment of personal intention, and they are, as such, neither "common" nor "general."

But to what end or ends is the imaginative instrumental? To two ends; namely, to the theoretical end which is "complete knowledge," and to the practical end which is the "good," "results which may in some way minister to the advancement of appetite, affection, disposition, conduct," "satisfactions or working practical effects." Hence arises the question whether these two ends exhaust the cases, "whether there are other types of apprehension which either set up still further ends, or in some way reduce or reconcile the duality disclosed by these two?" To this question Baldwin replies, "There is a type of imaginative cognition, I wish at once to say, that does not allow of description under either of the two foregoing headings; a type which is motived not by the interest of completeness of knowledge or thought, nor yet by the interest of seeking satisfactions or working practical effects. There is a way of treating a content, usually and properly called 'esthetic,' that we may describe as both *over-logical* and *over-practical*, as not being strictly either of these, although involving both of them" (p. 13). "The outcome of our investigation is that in the esthetic mode of experience so defined, we have the only inkling of the way that the self-reality of inner control which is the postulate of the

practical and the worthful, and the thing-reality of external control which is the presupposition of knowledge and truth, *can in the process of experience come together after having fallen apart* in the development of cognition."

The last statement may be regarded as the main thesis of this third volume. To establish it involves an investigation of interests, an answer to the questions why we are interested in the objects of practical knowledge and in those of theoretical knowledge. The author's answer to these two questions is one and the same. We are interested in these two types of knowledge at bottom because of a profound esthetic impulse which finds satisfaction now in practical knowledge and now in theory. Hence the fundamental categories of the ethico-political consciousness and the fundamental categories of the scientific consciousness, alike, are esthetic. Both types of consciousness and experience present a unity in manifoldness of the esthetic type. Both are comprehended in a whole beautiful which is known in "contemplation." "The object of contemplative interest is not only an object, but an object that embodies and completes the self. The self is realized in it, and the experience becomes one that may be called absolute in certain well-defined senses." This experience, however, is never comparable to the Hindu Nirvana, "it is never a state of pure absorption and loss of personal consciousness in an 'ecstasy' of feeling. . . . It absorbs the object in the self, which becomes the principle of realization of both knowledge and will; at the same time that, as being a state in which the dualistic terms are merged, it has the immediateness of feeling." This esthetic contemplation is elsewhere spoken of as a type of imagination along with the theoretic and practical. Again, the terms intuition and sympathy are used to express aspects of the esthetic experience, as the terms completeness, perfection, and uniqueness express aspects of the beautiful whole. In the beautiful the self finds its final fulfilment and profound satisfaction.

It is the self-interest which gives *a priori* universality and unconditionally imperative character to the moral obligations which get their content from the experienced consequences of conduct (pp. 102 ff.). And it is the same self-interest which through imitation and self-exhibition gives significance to the various forms of art (pp. 225 f. et a.). In a section on the correlation of practical and theoretical knowledge, the text maintains that the practical and the theoretical are, alike, knowledge, and that therefore the fundamental logical considerations that apply to the one apply to the other also (pp. 106 ff.). When we sympathetically contemplate the good we find it to be the complement and fulfilment of the self just as when we thus esthetically contemplate the true.

In a section on "Modes of Immediacy," three distinct types are distinguished (p. 231), namely, the immediacy of primitive sentience, "pure experience" or feeling; the immediacy of completion or "transcendence" where purposes are fulfilled and the mystic sinks into his ecstasy; and the immediacy of "reconciliation" where the dualisms of truth and value, means and end, self and the external world, terminate,—an esthetic and

supra-rational coincidence of opposites such as Nicholaus of Cusa perhaps regarded as our final intuition of the real. It is of course the latter immediacy which Professor Baldwin regards as the reconciliation of the practical and the theoretical, the inner and the outer, the worthful and the truthful. This leads to the definition of esthetic immediatism,—what Baldwin once called estho-monism,—and the invention of the name Pancalism for the doctrine that "esthetic experience presents the profounder significance of which truth and utility (the theoretical and practical) are partial and immature factors. The intuition of reality reached in esthetic contemplation preserves all the meaning of fact or truth except its externality to experience, and all that of use or worth except its subjectivity in experience; thus essentially removing from the constitution of the real the opposition of inner and outer, subject and object (p. 256). "Again, it preserves all the meaning of what is established, the 'is,' except its mere actuality, and all that of the desirable, the 'ought,' except its mere ideality; thus showing that the dualism of the actual and ideal is not intrinsic to the real" (p. 257).

Such, in outline, is the doctrine of this volume. The intellectual project of the work is most interesting. We have great confidence in Baldwin's viewpoint, his way of reaching a statement of the problems of contemporary philosophy, and also in his genetic method of investigation. But we find serious difficulties in some of his solutions. As I have elsewhere indicated, it seems to me that the dualism of inner and outer controls, the ego and the external world, is a presupposition of Baldwin's entire treatment of cognition. He has not shown how this dualism results from the growth of the cognitive functions. And, consequently, his esthetic experience, like Kant's purposive *Urtheilskraft,* can have only phenomenal validity.

Again, if esthetic contemplation is a form of imaginative cognition, and the imaginative is always instrumental (as the work maintains), it would seem that the dual end to which it ought to be instrumental, since it in a way solves and comprehends the practical and theoretical forms of imagination, is practical and theoretical knowledge. In other words, the limitations which the book ascribes to these two forms of knowledge ought to be predicable of esthetic contemplation. It can be said in reply that the latter is a form of immediacy, while practical and theoretical knowledge are mediate. So the author teaches. But, as a form of immediacy, it is just feeling, a sense of satisfaction or wholeness, and from this point of view, the author's esthetic absolute differs only in name from the absolute of Mr. Bradley,—unless Baldwin means that this esthetic immediacy is merely the various immediate experiences of different human beings, in which case we have a reassertion of Romanticism. Esthetic contemplation is (1) a "type of imaginative cognition," (2) a "mode of immediacy," and (3) a sort of mystical union of opposites possessing what might be called ontological validity; and we find it impossible to unite these characterizations in an intelligible concept.

Again, the distinction between what the book calls practical knowledge

and theoretical knowledge is most vague. The characterization of the former is to the present writer exceedingly unsatisfactory. Does Baldwin mean that *practise* can be reduced to terms of knowledge of practise? Is the whole world of human action, the world of rights and obligations, of freedom and responsibility, a mere epiphenomenon of esthetic immediacy? Perhaps the fourth volume will discuss this question and answer it, but we find the whole treatment of "The Logic of Practice," devoted as it is to the subject of affective logic in the sense of Ribot, to be inadequate as a treatment of the logic of ethico-political values.

The entire subject of religious experience is reserved for the fourth volume, in which the meaning of the term reality is to be further investigated. We shall await with interest the appearance of that work. The method of the entire treatise is genetic, and hence different from the dialectic method of such a work as Bradley's "Appearance and Reality." Where the latter demolishes the pretensions of the intellect by an inexorable logic, Baldwin's work simply attempts to show that for the knower the results of his theoretical and practical judgments are not *absolutely* valid. Have we, then, in the work, another of the creations of the romantic spirit in modern philosophy? Assuredly, it will not be the least of them in importance.

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History as Past Ethics: An Introduction to the History of Morals. PHILIP VAN NESS MYERS. Boston: Ginn & Company. 1913. Pp. 387.

The author's thesis that the philosophy of ethics "must be based on a knowledge of the facts of the moral life of the race in all the various stages of the historic evolution" is in praiseworthy harmony with the sociological attitude of to-day, but it is to be doubted whether this book is a successful presentation of that thesis. The author is a veteran writer of historical text-books, and does not seem to possess a really sociological point of view. On examination the book turns out to be based not upon the facts of the moral life, that is, upon the ethical practises of the societies discussed as related in the literature of historians, travelers, and anthropologists—the method which has made the works of Hobhouse and Westermarck so illuminating,—but rather upon the moral codes and sacred books and philosophic theorizing of the ethicists of the ages. The book is, therefore, not a real history of morals, if by morals we mean social practise into which enters the element of choice and approval, but a compilation and summary of the moral beliefs as codified by the articulate elements of the societies. It is, in other words, a history of the moralists, rather than of morals. A true history of morals would involve much more than this; it would have to include a genetic account of the ideals in relation to the social complex, the mass realization of those ideals and their slow alteration through the ages.

The author's point of view is too static,—one might say, too ethical. For he insists upon interpreting the ideals, beliefs, and prejudices of the

ancient Egyptians, Babylonians, Chinese, Hindus, Greeks, Romans, and modern Europeans, all equally from the standpoint of the Spencerian Protestant. His approving quotation from Petrie (p. 39), to the effect that the virtues and vices of ancient Egyptian morality belong "far more to the tone of Chesterfield and Gibbon than to that of Kingsley and Carlyle" is perhaps as delicious a remark as can be found in the whole range of ethical literature. The book is too much colored by this intellectualism which has made the study of ethics in the past a branch of logic rather than of sociology. Surely a statement that "man is by nature a moral being" is not illuminating from either an historical or sociological point of view, and to say that "the discontinuance of cannibalism may with little hesitation be ascribed to the growth and refinement of the moral feelings" is about as valuable as to say that the discontinuance of the protectionist policy may be with little hesitation ascribed to the removal of the tariff on imports. These facile causal hiatuses are not untypical of the school in which the author was bred, but that was a school that was not particularly successful in deriving theories from facts. Nothing, of course, could be more important than a simply presented and truly inductive historical study of morals, but it must be written from a point of view which sees morality as a general function of the social complex rather than as an inherent spiritual force.

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JOURNALS AND NEW BOOKS

REVUE NÉO-SCOLASTIQUE DE PHILOSOPHIE. May, 1913. *La philosophie syndicaliste et le mythe de la grève générale* (pp. 129–163): F. DE VISSCHER. – Socialism and democracy are diametrically opposed. The inspiring idea of socialism is a philosophy of action akin to Bergson's thought. *Roger Bacon et la composition des trois Opus* (pp. 164–180): P. MANDONNET. – The "Opus minus" and the "Opus tertium" are schemes of works which were never completed. Bacon has drawn from them many elements which have gone to constitute the "Opus majus." *La démonstration métaphysique du libre arbitre* (pp. 181–204): P. DE MUNNYNCK. – The faculty of will possesses, with regard to the acts dealing with a definite object, a power of determination which is called freedom. *La préparation scientifique nécessaire à l'étude de la cosmologie* (pp. 205–217): J. LEMAIRE. – For the man who wants to specialize in cosmology, a thorough study of the natural sciences is a necessary preparation. *Notes et discussions. Comptes-rendus.* Mgr. Farges, *La Philosophie de M. Bergson:* L. DU ROUSSAUX. Horten, *Die speculative und positive Theologie des Islam nach Razi und ihre Kritik durch Tusi:* J. FORGET. Marius Latour, *Premiers principes d'une théorie générale sur les émotions:* J. LEMAIRE. Reinhold Geijer, *Die Situation auf dem psychologischen Arbeitsfeld:* FR. FRANSEN. J. V. de Groot, *Eenige beschouwingen over "Lapsing intelligence":* FR. FRANSEN. C. Radelet, *Études philosophiques de théodicée:*

J. FORGET. G. di Ruggiere, *La filosofia contemporanea:* F. PALHORIÈS. E. Gilsn, *Index scolastico-cartésien:* M. DE WULF. P. Coffey, *The Science of Logic:* M. DE WULF. *Chronique. Sommaire idéologique des ouvrages et revues de philosophie.*

THE PHILOSOPHICAL REVIEW. May, 1913. *New Aims and Tendencies in Psychology* (pp. 251–264): FELIX KRUEGER.—To understand the study of modern psychology, note must be taken chiefly of America and Germany. The relation of psychology to other sciences must be determined, chiefly the natural and humanistic, with, perhaps, the shattering of conventional divisions of scientific work. *Mind as Form and as Activity* (pp. 265–283): GEORGE P. ADAMS.—The relational theory of consciousness, descriptive of both absolute idealism and new realism, issues in the view of mind as form according to which consciousness is cognitive, with a resulting intellectualism. The view of mind as activity is believed to be more adequate. *Methodological Realism* (pp. 284–303): G. A. TAWNEY.—Methodological realism—methodological because it declines to give a metaphysics—can not give an empirical and thoroughgoing account of the facts of experience without modifying its two cardinal theses, the relational theory of consciousness and the externality of relations. *Discussion: Duration and Value* (pp. 304–306): JARED S. MOORE. *In Reply to Professor Schaub* (pp. 306–307): ELLEN BLISS TALBOT. *Reviews of Books:* B. Bosanquet, *The Principle of Individuality and Value:* ERNEST ALBEE. James Mark Baldwin, *Thought and Things. A Study of the Development and Meaning of Thought, or Genetic Logic,* Vol. III.: EDWARD L. SCHAUB. Max Wundt, *Geschichte der griechischen Ethik:* WARNER FITE. Otto Gilbert. *Griechische Religionsphilosophie:* G. S. BRETT. *Notices of New Books. Summaries of Articles. Notes.*

Arnold, G. F. Psychology Applied to Legal Evidence and Other Constructions of Law. Second Edition. Calcutta: Thacker, Spink, and Company. 1913. Pp. xi + 607. 12s.

Bergman, Hugo. Das Unendliche und die Zahl. Halle: Verlag von Max Niemeyer. 1913. Pp. vii + 88.

Carus, Paul. The Principle of Relativity in the Light of the Philosophy of Science. Chicago and London: The Open Court Publishing Company. 1913. Pp. 105. $1.00.

NOTES AND NEWS

LETTER FROM PROFESSOR URBAN

TO THE EDITORS OF THE JOURNAL OF PHILOSOPHY, PSYCHOLOGY, AND SCIENTIFIC METHODS:

PROFESSOR SHELDON'S recent letter[1] deserves, in my opinion, earnest consideration. May I be permitted to say a word in favor of his proposal to simplify the statement of the Problem of Values, as presented for discussion at the next meeting of the American Philosophical Association?

[1] This JOURNAL, Vol. X., page 587.

And first I wish to speak of a practical matter touched upon in the letter. With Professor Sheldon I am inclined to think that the tendency toward over-organization that has shown itself recently in the proceedings of the Association is likely to hamper, rather than free, discussion. Doubtless there must be others who, like myself, have hoped to take part in the coming discussion, but have felt themselves embarrassed in seeking to express the results of their investigations in conformity with a scheme of thought foreign to their own method, and have found the results losing in naturalness and lucidity. It is even possible that there have been those in the past who have felt themselves excluded by the very definition of the problem.

For the devotion and energy of the committee all of us have, I am sure, nothing but thanks, and I wish personally to acknowledge my own indebtedness for many suggestions received from the formulation of the Problem of Values as they have stated it. I am also well aware that this formulation was meant to be mainly a point of departure for the discussion. At the same time, I can not help feeling that if we adhere too strictly to the programme, as at present formulated, the inevitable result will be to narrow the discussion in such a way that we shall finally end in going over the old ground of recent debates, and the distinctive problem of values itself, from the discussion of which some of us have looked for much light, will be relegated to the background.

With Professor Sheldon's plea for a subordination, if not elimination, of the epistemological problem, I am therefore in full accord, and I am glad to confess myself one "of the increasing number of philosophers who feel that it is not of great importance whether reality is subjective or objective, but that it is of surpassing interest to know what is the structure of reality." Precisely in this matter of "values" it seems to me there are but two ways of approach that are really significant and fruitful, and which may be expected to throw light upon this question of structure, *i. e.*, the scientific or psychological and the ontological, as Professor Sheldon calls it. The first of these has already developed its own methods and technique and may be said without serious risk of contradiction to have been fruitful in its own limited field. The second, in many respects more important problem, has become increasingly prominent in recent discussion, but there is still a conspicuous lack either of clear definition or consistent method. The attempt to bring order into this confusion may well occupy the energies of one meeting of the Association.

With this end in view Professor Sheldon has proposed to amend the programme in such a way as to bring out what seem to me to be the most important preliminary questions in this field. In order not to complicate the question further by presenting an analysis of my own, I prefer to subscribe myself in principle to his statement of the problem, and to express the hope that it will receive serious consideration.

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